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Inada Conditions and the Law of Diminishing Returns

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Abstract

Inada (1963) provided properties of the production function that are useful in the study of economic growth. Shephard (1970a) provided an axiomatic approach to the study of production theory. He applied these axioms to give a formal statement of the law of diminishing returns [(Shephard, 1970b)]. In this paper we demonstrate that the Inada conditions and the law of diminishing returns, as articulated by Shephard, are fundamentally inconsistent. Thus one is forced to make a choice between the two models when studying productivity and growth.

Key words: production; growth

JEL classification: D2; O4

1. Introduction

In this short note we examine the Inada (1963) and other related restrictions on the production function that are employed in both neoclassical and endogenous growth theory. The analysis is carried out in the context of the axiomatic approach to production theory advanced by Shephard (1970a). [See also Dyckhoff (1983).]

Let $R_+ = \{x \in R : x \geq 0\}$ be the set of nonnegative real numbers and let $R_{++} = \{x \in R : x > 0\}$ be the set of positive real numbers. Cartesian products are written as $R_+^2 = R_+ \times R_+$ and $R_{++}^2 = R_{++} \times R_{++}$. Write the production function as

$$F : R_+^2 \to R_+ \quad \text{with image} \quad Y = F(K, L),$$

where $Y \in R_+$ is output, $K \in R_+$ is capital, and $L \in R_+$ is labor. Following Barro and Sala-i-Martin (1995), we make the following set of assumptions.

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F.1 F is twice continuously differentiable everywhere on R^2_{++}.

F.2 $F(0, 0) = 0$.

F.3 $F(\lambda K, \lambda L) = \lambda F(K, L)$ for all $K, L, \lambda \in R_+$. (constant returns to scale (CRS))

F.4 For all $K, K', L, L' \in R_+$ if $K' \geq K$ and $L' \geq L$ then $F(K', L') \geq F(K, L)$.
(K and L are strongly disposable)

Assumptions F.1 – F.4 are used in all of our theorems below. We will also adopt the axiomatic framework of Shephard (1970b). The basic set of axioms is presented here.

A.1 $F(0, 0) = 0$.

A.2 $F(K, L) < \infty$ for all $(K, L) \in R^2_+$.

A.3 For all $K, K', L, L' \in R_+$ if $K' \geq K$ and $L' \geq L$ then $F(K', L') \geq F(K, L)$.
(K and L are strongly disposable)

A.4 For all $K, L \in R_+$ if $F(\lambda K, \lambda L) > 0$ then $F(\lambda K, \lambda L) \to \infty$ as $\lambda \to \infty$.

A.5 $F(K, L)$ is upper semicontinuous.

A.6 $F(K, L)$ is quasiconcave.

The first five axioms, A.1 – A.5, are weaker than the conditions imposed on the production function in F.1 – F.4. Note, for example, that the obtainability axiom, A.4, is implied by assumption F.3 (constant returns to scale). [Moreover, assumptions F.2 and F.4 are identical to assumptions A.1 and A.3, respectively, but this redundancy is harmless.]

In the next section we will state eight additional properties that the production function might possess; these properties are related to each other in the theorems. Properties (1), (3), (5), and (7) presented below are Inada conditions.

2. Inada Conditions, Essential and Limitational Inputs

An Inada condition often used in neoclassical growth theory is given by:

$$\text{For all } L \in R_{++}, \lim_{K \to \infty} F_K(K, L) = 0, \quad [I_{K \to \infty}]. \tag{1}$$

where $F_K(K, L)$ denotes the marginal product of capital. We say that *labor is essential* if the following condition holds.

$$\text{For all } K \in R_+, F(K, 0) = 0. \tag{2}$$

The other Inada condition for capital is:

For all $L \in R_{++}, \lim\limits_{K \to 0} F_K(K,L) = \infty, \quad [I_{K \to 0}]$. (3)

We say that *capital is limitational* if for all $K \in R_{++}, \lim_{L\to\infty} F(K,L) < \infty$. Thus, we say that capital is *not* limitational if the following condition holds.

For some $K \in R_{++}, \lim\limits_{L \to \infty} F\ (K,L) = \infty.$ (4)

There are two corresponding Inada conditions for labor. The first of these is:

For all $K \in R_{++}, \lim\limits_{L \to \infty} F_L(K,L) = 0$, $[I_{L\to\infty}]$. (5)

Analogous to (2) we say that *capital is essential* if the following condition holds.

For all $L \in R_{+}, F(0,L) = 0.$ (6)

The other Inada condition for labor is:

For all $K \in R_{++}, \lim\limits_{L \to 0} F_L(K,L) = \infty$, $[I_{L\to 0}]$. (7)

We say that *labor is limitational* if for all $L \in R_{++}, \lim_{K\to\infty} F(K,L) < \infty$. Thus, we say that labor is *not* limitational if the following condition holds.

For some $L \in R_{++}, \lim\limits_{K \to \infty} F(K,L) = \infty.$ (8)

Conditions (5) – (8) can be obtained from conditions (1) – (4) by simply interchanging the roles of capital and labor. In what follows, we will state our theorems in pairs; this will exploit the symmetry between capital and labor in the conditions (1) – (8).

3. Main Results

Theorem 1a: Assume F.1 – F.4. The first Inada condition $[I_{K\to\infty}]$ (1) implies that labor is essential (2).

The proof of this theorem is found in Barro and Sala-i-Martin (1995, p.52). For the reader's convenience we reproduce the proof here.

Proof 1a: Suppose $Y \to \infty$ as $K \to \infty$. Then

$$\lim_{K\to\infty} \frac{F(K,L)}{K} = \lim_{K\to\infty} F_K(K,L) = 0$$

where the first equality follows from L'Hôpital's rule and the second equality follows from (1). When Y is bounded as $K \to \infty$ we still get the result that

$$\lim_{K\to\infty} \frac{F(K,L)}{K} = 0.$$

By constant returns to scale,

$$\lim_{K\to\infty} \frac{F(K,L)}{K} = \lim_{K\to\infty} F(1, L/K) = F(1, 0)$$

and hence $F(1, 0) = 0$. Using CRS once again, $F(K, 0) = K\,F(1, 0) = 0$.

Theorem 1a has a counterpart for capital, namely,

Theorem 1b: Assume F.1 – F.4. The third Inada condition $[I_{L\to\infty}]$ (5) implies that capital is essential (6).

Next, we focus on Shephard's (1970b) formulation and proof of the law of diminishing returns. Färe (1980) showed that A.6 is not required for the proof of the law of diminishing returns. However, Shephard invoked the assumption that the efficient subsets are bounded. To clarify, first define the input requirement set, $\mathcal{L}(Y) = \{(K,L):\ F(K,L)\ \geq Y\}$ for each $Y \in R_+$. Then the efficient subset of $\mathcal{L}(Y)$ is

$$EFF\ \mathcal{L}(Y) = \left\{ (K,L): \begin{array}{c} \text{(i) } F(K,L) \geq Y \text{ and} \\ \text{(ii) if } (K',L') \leq (K,L) \text{ and } (K',L') \neq (K,L) \\ \text{then } F(K',L') < Y \end{array} \right\}.$$

For each Y, it is assumed that $EFF\ \mathcal{L}(Y)$ is a bounded set. Formally,
[*] $EFF\ \mathcal{L}(Y)$ is a bounded set for all $Y \in R_+$.

Shephard (1970a) justifies this assumption as one that is "… imposed as an obvious physical fact that no output rate is attained efficiently (in a technical sense) by an unbounded input vector," (p.15). The Cobb-Douglas production function does not have bounded efficient subsets "… and hence is not a valid production function over the entire domain …" (p.57).

Under the above axioms, including boundedness of the efficient subsets, Shephard (1970b) proves the following theorems. This constitutes his formal statement of the law of diminishing returns.

Theorem 2a: Assume F.1 – F.4, A.1 – A.6, and [*]. Then labor is essential (2) if and only if labor is limitational, i.e., $\neg(8)$.

Proof 2a: See Shephard (1970b). A simplified proof that invokes the stronger assumptions made in this paper (viz. CRS) is given in the Appendix.

The analogous theorem for capital is:

***Theorem 2b*:** Assume F.1 – F.4, A.1 – A.6, and [*]. Then capital is essential (6) if and only if capital is limitational, i.e., $\neg(4)$.

Färe (1980) weakened Shephard's boundedness assumption in his proof of the law of diminishing returns. The condition that he imposed was that if the infimum of the distance from the efficient subset to the capital or labor axis is zero, then the infimum is a minimum. The economic meaning of this is that for positive output, complete substitution of capital for labor (or labor for capital) is not possible by using an unbounded amount of capital (or labor). Again, the Cobb-Douglas production function does not meet this condition.

Combining Theorems 1a and 2a yields

***Theorem 3a*:** Assume F.1 – F.4, A.1 – A.6, and [*]. Then the first Inada condition $[I_{K\to\infty}](1)$ and the condition that labor is not limitational (8) cannot hold simultaneously.

Proof 3a: $(1) \Rightarrow (2) \Rightarrow \neg(8)$ by Theorems 1a and 2a. The contrapositive statement is $(8) \Rightarrow \neg(1)$. Thus, (1) and (8) cannot hold simultaneously.

Combining Theorems 1b and 2b yields

***Theorem 3b*:** Assume F.1 – F.4, A.1 – A.6, and [*]. Then the Inada condition $[I_{L\to\infty}](5)$ and the condition that capital is not limitational (4) cannot hold simultaneously.

The next pair of theorems characterizes the logical relation between the other two Inada conditions and limitational inputs.

***Theorem 4a*:** Assume F.1 –F.4 and A.1 – A.6. Then the Inada condition $[I_{K\to 0}](3)$ implies that capital is not limitational (4).

Proof 4a:

Choose any $K \in R_{++}$. Then

$$\begin{aligned}\lim_{L\to\infty} F(K,L) &= K \lim_{L\to\infty} \frac{F(K/L,1)}{K/L} \quad \text{(using F.3)} \\ &= K \lim_{K\to 0} \frac{F(K/L,1)}{K/L}.\end{aligned} \tag{9}$$

If $F(0,1) > 0$ then the last term in (9) is ∞ and thus,

$$\lim_{L\to\infty} F(K,L) = \infty.$$

On the other hand, if $F(0,1)=0$ then we can apply L'Hôpital's rule to (9) to get

$$\begin{aligned} K\lim_{K\to 0}\frac{F(K/L,1)}{K/L} &= KL\lim_{K\to 0}F_K(K/L,1) \\ &= K\lim_{K\to 0}F_K(K,L)\ \text{(using F.3)} \\ &= \infty.\ [\text{using (3)}] \end{aligned}$$

Similarly, we have

Theorem 4b: Assume F.1 – F.4 and A.1 – A.6. Then the Inada condition $[I_{L\to 0}]$ (7) implies that labor is not limitational (8).

We are now in a position to state the following result.

Theorem 5a: Assume F.1 – F.4, A.1 – A.6, and [*]. Then the Inada condition $[I_{K\to 0}]$ (3) and the Inada condition $[I_{L\to\infty}]$ (5) cannot hold simultaneously.

Proof 5a: By Theorem 4a, (3) implies (4). By Theorem 3b, (4) and (5) cannot hold simultaneously. So $(3)\Rightarrow(4)\Rightarrow\neg(5)$. Also, $(5)\Rightarrow\neg(4)\Rightarrow\neg(3)$. So (3) and (5) cannot hold simultaneously.

This theorem illustrates the underlying conflict between the law of diminishing returns as articulated by Shephard and the Inada conditions applied to both capital and labor. The analogous theorem is

Theorem 5b: Assume F.1 – F.4, A.1 – A.6, and [*]. Then the Inada condition $[I_{K\to\infty}]$ (1) and the Inada condition $[I_{L\to 0}]$ (7) cannot hold simultaneously.

A final pair of theorems is also implied by the previous results.

Theorem 6a: Assume F.1 – F.4, A.1 – A.6, and [*]. Then the Inada condition $[I_{K\to 0}]$ (3) and the condition that capital is essential (6) cannot hold simultaneously.

Proof 6a: By Theorem 4a, (3) implies (4). By Theorem 2b, (6) if and only if $\neg(4)$. Thus $(3)\Rightarrow(4)\Rightarrow\neg(6)$. Also, $(6)\Rightarrow\neg(4)\Rightarrow\neg(3)$.

Theorem 6b: Assume F.1 –F.4, A.1 –A.6, and [*]. Then the Inada condition $[I_{L\to 0}]$ (7) and the condition that labor is essential (2) cannot hold simultaneously.

4. Conclusion

It follows from these results that one must make a choice between these two approaches to production theory when studying topics in productivity and growth. Further research, however, could mollify this trade-off. The task would be to preserve the central propositions of growth theory in the presence of the law of diminishing returns as characterized by Shephard.

Appendix

Proof 2a: Recall that

$$\pounds(Y) = \{(K,L) : F(K,L) \geq Y\}.$$

Our assumptions imply the following properties of $\pounds(Y)$, EFF $\pounds(Y)$, and the closure of EFF $\pounds(Y)$.

(i) $\pounds(Y) = Y\pounds(1)$ and $EFF\ \pounds(Y) = Y\ EFF\ \pounds(1)$ (by F.3, CRS)

(ii) $\pounds(Y) = EFF\ \pounds(Y) + R_+^2$ (by F.4, strong disposability of inputs)

(iii) $EFF\ \pounds(Y) \neq \phi$ (by F.1, F is continuous and hence $\pounds(Y)$ is closed)

(iv) $\pounds(Y) = Y\ EFF\ \pounds(1) + R_+^2$ (by (i) and (ii))

(v) $\pounds(Y) = Y\ \overline{EFF\ \pounds(1)} + R_+^2$, where $\overline{EFF\ \pounds(1)}$ is the closure of $EFF\ \pounds(1)$ ($\pounds(Y)$ is closed)

(vi) $\overline{EFF\ \pounds(Y)}$ is compact (since $EFF\ \pounds(Y)$ is bounded and closed) and nonempty (by (iii))

(vii) $\overline{EFF\ \pounds(Y)} = Y\ \overline{EFF\ \pounds(1)}$ (by(i))

Now define

$$L^m(Y) = \min_{K,L}\{L : (K,L) \in \overline{EFF\ \pounds(Y)}\}. \tag{10}$$

This minimum exists since $\overline{EFF\ \pounds(Y)}$ is compact and nonempty.

Now assume that labor is essential, i.e., assume (2). Since $F(K,0) = 0$ it must be the case that $L^m(K,Y) > 0$ for all $K \in R_{++}$ and $Y \in R_{++}$. Moreover, by CRS,

$$L^m(Y) = YL^m(1)$$

since

$$\begin{aligned}
L^m(Y) &= \min_{K,L}\left\{L : (K,L) \in \overline{EFF\ \ \pounds(Y)}\right\} \\
&= \min_{K,L}\left\{L : (K,L) \in Y\overline{EFF\ \ \pounds(1)}\right\} \quad \text{(by (vii))} \\
&= Y \min_{K/Y,\, L/Y}\left\{\frac{L}{Y} : \left(\frac{K}{Y}, \frac{L}{Y}\right) \in \overline{EFF\ \ \pounds(1)}\right\} \\
&= YL^m\ (1).
\end{aligned}$$

Fix a value of labor at any arbitrary value, $L = \overline{L}$. By choosing a large enough output level, $\hat{Y}$, we can force $L^m(\hat{Y}) = \hat{Y} L^m(1) > \overline{L}$. From this and the definition of $L^m(\hat{Y})$ it follows that

$$\mathcal{L}(\hat{Y}) \cap \left\{ (K, L) : K \geq 0,\ L \leq \overline{L} \right\} = \phi ,$$

i.e., $\hat{Y}$ is unobtainable when $L \leq \overline{L}$ and thus labor is limitational, i.e., we have $\neg(8)$.

To prove the converse, suppose L is not essential. Then there exists an input vector, $(K, 0)$, such that $F(K, 0) > 0$. By F.3 (CRS), $F(\lambda K,\ \lambda 0) = \lambda F(K, 0) > 0$ and thus

$$\lim_{\lambda \to \infty} F(\lambda K,\ \lambda 0) = \infty ,$$

and L is not limitational. So, if labor is limitational, i.e. $\neg(8)$, then the labor is essential (2). ∎

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