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A Note on Second-Order Conditions for Maximizing Monopolist's Revenue and a Quantity-Setting Symmetric Duopoly

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1. Motives and Results

Assume throughout that the market demand curve is downward-sloping and supported by a twice-continuously differentiable demand function f, whose inverse function is denoted by g. If the market demand is to be satisfied by a monopolist, her (total) revenue is $TR = f(P)P = g(Q)Q$. Most students in principles of economics course know that choosing P to maximize TR is equivalent to choosing Q to maximize TR. What are not mentioned in textbooks for (intermediate) microeconomics or introductory mathematical economics are two technical issues:

(1) Can we say that the second-order condition of "choosing P to maximize $f(P)P$" is satisfied *if and only if* that of "choosing Q to maximize $g(Q)Q$" is satisfied?

(2) Duopoly naturally follows monopoly in course coverage. If the second-order condition of "choosing Q to maximize $g(Q)Q$" is satisfied, must the second-order condition of "for each i of {1, 2} and at each given q_j ($j \neq i$), choosing q_i to maximize $g(q_1 + q_2)q_i$" be satisfied? How about the converse?

Obviously, for any concave demand function ($d^2 f(P)/dP^2 \leq 0$), both the second-order condition of "choosing P to maximize $f(P)P$" and that of "choosing Q to maximize $g(Q)Q$" are satisfied. When the demand is strictly convex ($d^2 f(P)/dP^2 > 0$), we show that neither of these two second-order conditions implies the another.

As to (2), it is about maximizing monopolist's revenue and finding the

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revenue-maximizing output decision for each firm in a symmetric duopoly. [With a constant average and marginal cost $c > 0$, we can change the term revenue-maximizing to profit-maximizing.] We show that for strictly convex demand, the second-order condition of "choosing Q to maximize $g(Q)Q$" is satisfied *if and only if* the second-order condition of "for each i of {1, 2} and at each given q_j ($j \neq i$), choosing q_i to maximize $g(q_1 + q_2)q_i$" is satisfied. An interesting example concludes.

2. Revenue Maximization for a Monopolist

A standard and simplest way to teach revenue maximization starts with a linear demand, say $Q = f(P) := a - bP$ with $a > 0$ and $b > 0$. Here, $TR = f(P)P = aP - bP^2$, strictly concave in P, likewise for $TR = g(Q)Q = (aQ - Q^2)b^{-1}$. The intuitive explanation of why both second-order conditions (in revenue maximization) are satisfied is easy: $f(P)$ (*resp.* $g(Q)$) is linear in P (*resp.* Q) and negatively correlated with P (*resp.* Q). What if $f(P)$ (*resp.* $g(Q)$) is not linear in P (*resp.* Q)? The instinct tells us that for strictly concave demand function ($d^2 f(P)/dP^2 < 0$, hence $d^2 g(Q)/dQ^2 < 0$), both second-order conditions in (1) are satisfied. The following simple algebra tells it all. At P_0,

$d[f(P)P]/dP = f(P_0) + [df(P)/dP]P_0$ (with $df(P)/dP$ evaluated at $P = P_0$) and $d^2[f(P)P]/dP^2 = 2[df(P)/dP] + [d^2 f(P)/dP^2]P_0$.

Likewise, at Q_0,

$d[g(Q)Q]/dQ = g(Q_0) + [dg(Q)/dQ]Q_0$ (with $dg(Q)/dQ$ evaluated at $Q = Q_0$) and $d^2[g(Q)Q]/dQ^2 = 2[dg(Q)/dQ] + [d^2 g(Q)/dQ^2]Q_0$.

By $df(P)/dP < 0$ and $dg(Q)/dQ < 0$, we see that $d^2[f(P)P]/dP^2 < 0$ and $d^2[g(Q)Q]/dQ^2 < 0$ as long as $d^2 f(P)/dP^2 \leq 0$ (or $d^2 g(Q)/dQ^2 \leq 0$). So, we only have to worry about the case of strictly convex demand.

Consider the function $P = g(Q) := e^{-Q}$ defined for all non-negative Q. Such a strictly convex demand can be found in Forshner and Shy (2009) as well as Amir and Grilo (1999). At $P_0 > 0$,

$$d[f(P)P]/dP = -1 + ln(P_0) = 0 \quad \text{if} \quad P_0 = e^{-1}.$$
$$d^2[f(P)P]/dP^2 = -1/P_0 < 0.$$

At $Q_0 > 0$,

$d[g(Q)Q]/dQ = e^{-Q_0}(1 - Q_0)$ is positive if $Q_0 < 1$; zero if $Q_0 = 1$; negative if $Q_0 > 1$.

$d^2[g(Q)Q]/dQ^2 = e^{-Q_0}(Q_0 - 2)$ is negative if $Q_0 < 2$; zero if $Q_0 = 2$; positive if $Q_0 > 2$.

We see that $f(P)P$ is strictly concave in P yet $g(Q)Q$ is strictly concave only for Q in [0, 2]. In this case and thru either method, revenue is maximized at $P = e^{-1}$ (and $Q = 1$). The magnitude of $2[df(P)/dP]$ must have dominated that of $[d^2 f(P)/dP^2]P_0$ (for all P_0) while on the contrary, the magnitude of $2[dg(Q)/dQ]$ is less than that of $[d^2 g(Q)/dQ^2]Q_0$ for all $Q_0 > 2$.

The example given above shows that fulfilling the second-order condition of "choosing P to maximize $f(P)P$" does not imply that the second-order condition of "choosing Q to maximize $g(Q)Q$". To see why the converse does not hold, consider the iso-elastic demand function $Q = f(P) := P^{-2}$ defined for all $P > 0$. Note that TR is $g(Q)Q = Q^{0.5}$ defined for $Q > 0$ and that $d^2[g(Q)Q]/dQ^2 = -(1/4)Q^{-3/2} < 0$, yet $d^2[f(P)P]/dP^2 = 4P^{-3} > 0$. Having addressed issue (1), we can convince students why they should not give it up easily when the second-order condition at hand is not satisfied.

3. A Link with Revenue Maximization in a Quantity-Setting Duopoly

Suppose instead that two firms are competing in a quantity-setting symmetric duopoly. To find the Nash equilibrium, for each i of {1, 2}, firm i shall, at each *given* q_j (with $j \neq i$), choose q_i to maximize her profit $q_i g(q_1 + q_2)$. The first and second derivatives are respectively

$$d[q_i g(q_1 + q_2)]/dq_i = g(q_1 + q_2) + [dg(z)/dz]q_i \quad \text{and}$$
$$d^2[q_i g(q_1 + q_2)]/dq_i^2 = 2\,dg(z)/dz + [d^2 g(z)/dz^2]q_i \quad \text{where} \quad z := q_1 + q_2.$$

It is interesting to compare the last line with $d^2[g(Q)Q]/dQ^2 = 2[dg(Q)/dQ] + [d^2 g(Q)/dQ^2]Q_0$. With concave demand we see obviously $d^2[q_i g(q_1 + q_2)]/dq_i^2 < 0$ and $d^2[g(Q)Q]/dQ^2 < 0$. When demand is strictly convex, if $d^2[g(Q)Q]/dQ^2 < 0$, then $d^2[q_i g(q_1 + q_2)]/dq_i^2 < 0$ (for all i). The converse is also true although not so obvious. [It can be shown by considering sufficiently small q_j and recalling the continuity of functions.] Hence, with strictly convex demand, having the second-order conditions satisfied for each quantity-setting duopolistic firm is the same as having the second-order condition satisfied for the revenue maximization problem by choosing Q. This completes (2).

We conclude this note by showing, via the following example, that second-order conditions may fail *globally* in both settings yet monopolist's revenue can be maximized, so can the Nash equilibrium in the duopoly be found.

Recall $P = g(Q) := e^{-Q}$ defined for all $Q \geq 0$. We have shown that revenue can be maximized although $d^2[g(Q)Q]/dQ^2 < 0$ does not hold for all $Q > 0$. To find the Nash equilibrium, for each i of {1, 2}, firm i shall, at each *given* q_j (with $j \neq i$), choose q_i to maximize her profit $q_i g(q_1 + q_2) = q_i e^{-q_1 - q_2}$. Note that

$d[q_i g(q_1+q_2)]/dq_i$ is zero if $q_i = 1$; positive if $q_i < 1$; negative if $q_i > 1$.

$d^2[q_i g(q_1+q_2)]/dq_i^2 = e^{-q_1-q_2}(q_1-2)$ is negative if $q_1 < 2$; zero if $q_1 = 2$; positive if $q_1 > 2$.

Here, at each given q_j the function $q_i g(q_1+q_2)$ is not concave in q_i yet strict concavity holds in the neighborhood of $q_i = 1$ (i.e., the solution from the first-order condition). And this local maximum turns out to be the global maximum, yielding (1, 1) as the dominant strategy equilibrium as well as the Nash equilibrium.

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